DEPARTMENT OF COMMERCE AND LABOR

BULLETIN

OF THE

BUREAU OF STANDARDS

VOLUME 6

1909-10

DEPARTMENT OF COMMERCE AND LABOR • UNITED STATES OF AMERICA •

56

DEPARTMENT OF COMMERCE AND LABOR

BULLETIN

OF THE

BUREAU OF STANDARDS

S. W. STRATTON, Director

VOLUME 6

1909-10

WASHINGTON
GOVERNMENT PRINTING OFFICE
1910

THE MUTUAL INDUCTANCE OF TWO PARALLEL COAXIAL CIRCLES IN TERMS OF HYPERGEOMETRICAL SERIES

By Frederick W. Grover

In the Journal de Physique for 1901 (vol. 10, p. 33) there appeared a paper by E. Mathy entitled "Application des signes de Weierstrass à l'étude de l'énergie potentielle de deux courants circulaires parallèles d'intensité un," in which the author, by the introduction of the Weierstrassian notation, obtains the mutual induction of two parallel, coaxial circles in terms of hypergeometrical series, instead of the usual forms involving elliptic integrals.

His formula is

$$\frac{M}{4\pi}=\Bigg[[(b^2+r^2+r_1{}^2)^2+12r^2r_1{}^2]^{\frac{1}{4}}\left\{\frac{B}{3^{\frac{1}{4}}}F\left(-\frac{1}{12},-\frac{1}{12},\frac{1}{2},\frac{J-1}{J}\right)-\frac{A}{6.3^{\frac{3}{4}}}\sqrt{\frac{J-1}{J}}F\left(\frac{5}{12},\frac{5}{12},\frac{3}{2},\frac{J-1}{J}\right)\right\}$$

$$-\frac{b^2+r^2+r_1{}^2}{[(b^2+r^2+r_1{}^2)^2+12r^2r_1{}^2]^{\frac{1}{4}}}\left\{\frac{A}{3^{\frac{3}{4}}}.F\left(\frac{1}{12},\frac{1}{12},\frac{1}{2},\frac{J-1}{J}\right)+\frac{B}{6.3^{\frac{1}{4}}}\sqrt{\frac{J-1}{J}}F\left(\frac{7}{12},\frac{7}{12},\frac{3}{2},\frac{J-1}{J}\right)\right\}\Bigg]$$

where

$$A=1.311\ 028\ 777\ \ldots\ .$$
$$B=0.599\ 070\ 117\ \ldots\ .$$

$$\sqrt{\frac{J-1}{J}}=\frac{1-36\left(\frac{rr_1}{b^2+r^2+r_1{}^2}\right)^2}{\left[1+12\left(\frac{rr_1}{b^2+r^2+r_1{}^2}\right)^2\right]^{\frac{3}{2}}}$$

and the Gaussian notation for the hypergeometrical series has been adopted, viz—

$$F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha\beta}{1.\gamma}x + \frac{\alpha(\alpha+1).\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2).\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3 + \ldots$$

No numerical results were given by which one may judge of the degree of convergence of the series in a practical case, nor was, apparently, any comparison of the formula made with other expressions for the mutual inductance of two circles.

In order to obtain light on these questions, the numerical values of the mutual inductances of several pairs of circles were calculated by the above formula and compared with the results obtained by the use of the formulæ of Maxwell[1] and Nagaoka.[2] (See also This Bulletin, vol. 5, pp. 6, 8, 1908.) It was found, that only in the case where $\frac{J-1}{J}=0$ does Mathy's formula give correct results, and in that case M comes out with the negative sign. In the more general case where $\frac{J-1}{J}$ is not equal to zero, the formula of Mathy gives values in error by as much as 5 to 10 per cent.

I have therefore checked the derivation of the formula, with the result that a corrected expression was found which, if used within those limits in which it is rapidly convergent, agrees closely with the formulæ of Maxwell and Nagaoka; in other words, gives very accurate results. The derivation is given below, the notation being that of Mathy. Since the original derivation is very brief, and rather difficult to follow, the work will be given here somewhat in detail.

Let r and r_1 be the radii of the two parallel circles (Fig. 1), b the distance between their planes, and ds and ds', respectively, elements of their circumferences. To get the mutual inductance of the two circles we have then to find

$$M = \int\int \frac{ds\,ds'\cos\epsilon}{R}$$

[1] Elect. and Mag. Vol. **II**, § 701. [2] Phil. Mag., **6**, p. 19; 1903.

where ϵ is the angle between the radii vectores to ds and ds', and R is the distance between them, the integration being extended around both circumferences.

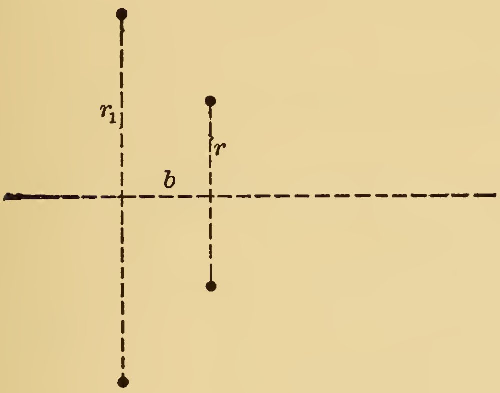

Fig. 1.

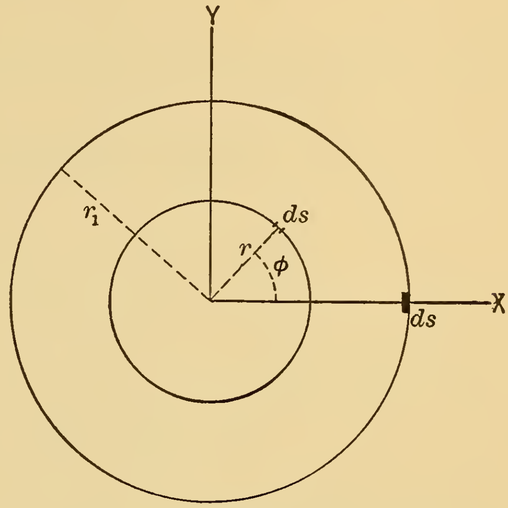

Fig. 2.

If we use polar coordinates (r, ϕ) and (r_1, ϕ_1), and, taking $\phi_1 = 0$, integrate around the circle of radius r, we find

$$dM = r_1\, d\phi_1 \int_0^{2\pi} \frac{r\, d\phi \cdot \cos \phi}{\sqrt{b^2 + r^2 + r_1^2 - 2rr_1 \cos \phi}}$$

Since the amount added to the mutual inductance by each element $ds' = r_1 d\phi_1$ is the same, we will have

$$M = 2\pi r r_1 \int_0^{2\pi} \frac{\cos \phi\, d\phi}{\sqrt{b^2 + r^2 + r_1^2 - 2rr_1 \cos\phi}}$$

Thus far, Maxwell's method has been used, but whereas he expressed the above elliptic integral in terms of the complete elliptic integrals F and E of the first and second kinds, Mathy next introduces the Weirstrassian function $\mathfrak{p}$ in order to obtain a series development of the integral.

Taking rectangular coordinates as shown in Fig. 2

$$\cos \phi = \frac{x}{r}, \quad d\phi = -\frac{dx}{\sqrt{r^2 - x^2}}$$

and we find

$$M = 4\pi r_1 \int_{-r}^{r} \frac{x\,dx}{\sqrt{(r^2 - x^2)(b^2 + r^2 + r_1^2 - 2r_1 x)}}$$

The polynomial under the radical is

$$2r_1(x - r)(x + r)\left(x - \frac{b^2 + r^2 + r_1^2}{2r_1}\right)$$

$$= 2r_1\left[x^3 - x^2\left(\frac{b^2 + r^2 + r_1^2}{2r_1}\right) - r^2 x + \frac{r^2(b^2 + r^2 + r_1^2)}{2r_1}\right]$$

To reduce this to the canonical Weierstrassian form

$$4(y - e_1)(y - e_2)(y - e_3) = 4y^3 - g_2 y - g_3$$

where $e_1 + e_2 + e_3 = 0$, we must make the coefficient of y^2 equal to zero. We put, therefore,

$$x = y + \frac{b^2 + r^2 + r_1^2}{6r_1}$$

and find

$$e_1 = -\left[\frac{b^2 + r^2 + r_1^2}{6r_1} - \frac{b^2 + r^2 + r_1^2}{2r_1}\right] = \frac{b^2 + r^2 + r_1^2}{3r_1}$$

$$e_2 = -\frac{b^2 + r^2 + r_1^2 - 6rr_1}{6r_1}$$

$$e_3 = -\frac{b^2 + r^2 + r_1^2 + 6rr_1}{6r_1} \quad \text{where } e_1 > e_2 > e_3$$

$$x = y + \frac{e_1}{2}$$

We may accordingly write

$$M = 2\pi\sqrt{2r_1} \int_{e_3}^{e_2} \frac{\left(y + \frac{e_1}{2}\right)dy}{\sqrt{(y - e_1)(y - e_2)(y - e_3)}}$$

$$= 4\pi\sqrt{2r_1}\left[\int_{e_3}^{e_2} \frac{y\,dy}{\sqrt{Y}} + \frac{e_1}{2}\int_{e_3}^{e_2} \frac{dy}{\sqrt{Y}}\right]$$

where

$$Y=\sqrt{4y^3-g_2y-g_3} \text{ and } y=\wp v$$

Now

$$v=\int_{e_3}^{e_2}\frac{dy}{\sqrt{Y}}=\int_{e_3}^{\infty}\frac{dy}{\sqrt{Y}}-\int_{e_2}^{\infty}\frac{dy}{\sqrt{Y}}=(2\omega+\omega')-(\omega+\omega')=\omega$$

ω and ω' being respectively the real and imaginary semi-periods of $\wp v$.

Consequently

$$\int_{e_3}^{e_2}\frac{ydy}{\sqrt{Y}}=-\zeta\omega=-\eta$$

and

$$M=4\pi\sqrt{2r_1}\left(\frac{e_1}{2}\omega-\eta\right) \tag{1}$$

This differs from the expression found by Mathy only in the algebraic sign of M. A similar expression was found by Nagaoka, who developed it in terms of the rapidly convergent q series of Jacobi. Mathy, on the other hand, referring to Halphen's "Traité des fonctions elliptiques," part 1, p. 313 (Gauthier-Villars, Paris, 1886), expands ω and η in terms of hypergeometric series involving the absolute invariant J. The hypergeometrical differential equations for ω and η are deduced by Halphen (p. 313) and the final result for ω is given in equation (21), p. 343. The next two equations in Halphen in which the hypergeometrical series written in the $[q_1, q_2, q_3; x]$ notation are expressed in the equivalent $F(\alpha, \beta, \delta, x)$ notation are incorrect, and their use by Mathy is responsible for part of the error in his formula for M.

These equations should read

$$\left.\begin{aligned}\left[\frac{1}{2}, 0, \frac{1}{3}; \frac{J-1}{J}\right]&=F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \frac{J-1}{J}\right)\\&=1+5\frac{2}{12^2}\frac{J-1}{J}+\frac{2^2}{12^4}\frac{5.13.17}{3}\left(\frac{J-1}{J}\right)^2+\ldots\\\left[-\frac{1}{2}, 0, \frac{1}{3}; \frac{J-1}{J}\right]&=F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}, \frac{J-1}{J}\right)\\&=1+\frac{2}{12^2}7.11\frac{J-1}{J}+\frac{2}{12^4}\frac{7.11.19.23}{3.5}\left(\frac{J-1}{J}\right)^2+\end{aligned}\right\} \tag{2}$$

where the $\frac{5}{12}$ replaces $\frac{1}{12}$, and the $\frac{11}{12}$ replaces $\frac{7}{12}$ in Halphen.

The relations connecting the absolute invariant J with the discriminant Δ and the invariants g_2 and g_3 of the Weierstrassian function $\wp v$ are

$$\Delta = g_2^3 - 27 g_3^2, \quad J = \frac{g_2^3}{\Delta}, \quad J - 1 = \frac{27 g_3^2}{\Delta} \tag{3}$$

Halphen puts

$$x = \omega \Delta^{\frac{1}{12}}, \quad y' = \eta \Delta^{-\frac{1}{12}} \tag{4}$$

From page 307 he finds

$$\left.\begin{aligned} \frac{dx}{dJ} &= -\frac{1}{2\sqrt{3}}(J-1)^{-\frac{1}{2}} J^{-\frac{2}{3}} y' \\ \frac{dy'}{dJ} &= \frac{1}{24\sqrt{3}}(J-1)^{-\frac{1}{2}} J^{-\frac{1}{3}} x \end{aligned}\right\} \tag{5}$$

and, introducing the auxiliary quantity,

$$y = \frac{1}{2\sqrt{3}}(J-1)^{-\frac{1}{2}} J^{-\frac{2}{3}} y' \tag{6}$$

he derives immediately two differential equations for x and y as functions of J.

The first of these (equation 54) is

$$J(1-J)\frac{d^2x}{dJ^2} + \frac{1}{6}(4-7J)\frac{dx}{dJ} - \frac{x}{144} = 0 \tag{7}$$

The equation (55) for y is incorrectly given by Halphen. It should read

$$J(1-J)\frac{d^2y}{dJ^2} + \left(\frac{5}{3} - \frac{19}{6}J\right)\frac{dy}{dJ} - \frac{169}{144}y = 0 \tag{8}$$

the $\frac{5}{3}$ replacing $\frac{5}{6}$ and the $\frac{169}{144}$ replacing $\frac{53}{48}$.

Both these equations are in the Gaussian form

$$x(x-1)\frac{d^2y}{dx^2} + [\gamma - (\alpha + \beta + 1)x]\frac{dy}{dx} - \alpha\beta y = 0$$

and particular solutions may readily be written down from any treatise which gives the various forms assumed by the hyper-

geometrical series. Since g_3 and Δ are positive, we see from (3) that $(J-1)$ is positive, and therefore $\frac{J-1}{J}<1$. The hypergeometrical series in $\frac{J-1}{J}$ will therefore converge.

Considering, first, equation (7) we find

$$\gamma=\frac{2}{3}, \quad \alpha=\frac{1}{12}, \quad \beta=\frac{1}{12}$$

Particular solutions are, therefore (see Weber, "Die Partiellen Differential-Gleichungen," II, p. 19, equations V)

$$x_1=J^{-\frac{1}{12}}F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \frac{J-1}{J}\right)$$

$$x_2=J^{-\frac{1}{12}}\sqrt{\frac{J-1}{J}}F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}, \frac{J-1}{J}\right)$$

and the complete solution is

$$x=aJ^{-\frac{1}{12}}F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \frac{J-1}{J}\right)+ibJ^{-\frac{1}{12}}\sqrt{\frac{J-1}{J}}F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}, \frac{J-1}{J}\right)$$

where a and b are arbitrary constants and $i=\sqrt{-1}$.

Accordingly, from (3) and (4)

$$g_2^{\frac{1}{4}}\omega=xJ^{\frac{1}{12}}=aF\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \frac{J-1}{J}\right)+ib\sqrt{\frac{J-1}{J}}F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}, \frac{J-1}{J}\right) \quad (9)$$

The constants are to be evaluated from the following considerations (Halphen, p. 342–343):

$$\lim_{J=1}(g_2^{\frac{1}{4}}\omega)=A\sqrt{2}=a$$

$$\lim_{J=1}\left(\sqrt{J-1}\cdot\frac{dx}{dJ}\right)=\frac{i}{2}b=-\frac{1}{2\sqrt{3}}\lim_{J=1}(\eta\Delta^{-\frac{1}{12}})$$

$$\lim_{J=1}(\eta\omega)=\frac{\pi}{4}$$

$$\therefore\ ib=-\frac{\pi}{4a\sqrt{3}}$$

The expression (9) becomes, therefore,

$$g_2^{\frac{1}{4}}\omega = A\sqrt{2}\,.\,F\left(\frac{1}{12},\frac{5}{12},\frac{1}{2},\frac{J-1}{J}\right)-\frac{B}{\sqrt{6}}\sqrt{\frac{J-1}{J}}\,.\,F\left(\frac{7}{12},\frac{11}{12},\frac{3}{2},\frac{J-1}{J}\right) \quad (10)$$

where A and B are Sterling's constants

$$A=\int_0^1\frac{dx}{\sqrt{1-x^4}}=1.311\ 028\ 777\ \ldots\ldots$$

$$B=\frac{\pi}{4A}=0.599\ 070\ 117\ \ldots\ldots$$

Correcting Halphen's result for ω by means of equations (2) it agrees with (10).

The derivation of the value of η is similar.

In equation (8) $\alpha=\frac{13}{12}$, $\beta=\frac{13}{12}$, $\gamma=\frac{5}{3}$. Accordingly

$$y=cJ^{-\frac{13}{12}}.F\left(\frac{13}{12},\frac{5}{12},\frac{3}{2},\frac{J-1}{J}\right)+dJ^{-\frac{7}{12}}(1-J)^{-\frac{1}{2}}F\left(\frac{7}{12},-\frac{1}{12},\frac{1}{2},\frac{J-1}{J}\right)$$

and from (6)

$$\left.\begin{aligned}y'=c2\sqrt{3}J^{\frac{1}{12}}\sqrt{\frac{J-1}{J}}F\left(\frac{13}{12},\frac{5}{12},\frac{3}{2},\frac{J-1}{J}\right)\\-2\sqrt{3}idJ^{\frac{1}{12}}F\left(\frac{7}{12},-\frac{1}{12},\frac{1}{2},\frac{J-1}{J}\right)\end{aligned}\right\}\quad(11)$$

or

$$\left.\begin{aligned}\eta g_2^{-\frac{1}{4}}=y'J^{-\frac{1}{12}}=2\sqrt{3}c\sqrt{\frac{J-1}{J}}F\left(\frac{13}{12},\frac{5}{12},\frac{3}{2},\frac{J-1}{J}\right)\\-2\sqrt{3}idF\left(\frac{7}{12},-\frac{1}{12},\frac{1}{2},\frac{J-1}{J}\right)\end{aligned}\right\}\quad(12)$$

To determine the constants we have

$$\lim_{J=1}(\omega\eta)=\lim_{J=1}(g_2^{\frac{1}{4}}\omega.g_2^{-\frac{1}{4}}\eta)=\frac{\pi}{4}$$

$$\lim_{J=1}\left(\frac{dy'}{dJ}\sqrt{J-1}\right)=c\sqrt{3}=\frac{1}{24\sqrt{3}}\lim_{J=1}\ (x)$$

from which $c=\dfrac{a}{36\sqrt{2}}, \quad -id=\dfrac{B}{2\sqrt{6}}$

Therefore, finally

$$\eta g_2^{-\frac{1}{4}}=\frac{B}{\sqrt{2}}F\left(\frac{7}{12}, -\frac{1}{12}, \frac{1}{2}, \frac{J-1}{J}\right)+\frac{A}{6\sqrt{6}}\sqrt{\frac{J-1}{J}}F\left(\frac{13}{12}, \frac{5}{12}, \frac{3}{2}, \frac{J-1}{J}\right) \quad (13)$$

The value of η was found by Mathy from the equations (4) and (5) without the employment of the auxiliary quantity y.

The differential equation for y' is, as was also found by Mathy (in Mathy's notation y' is y)

$$J(1-J)\frac{d^2y'}{dJ^2}+\left(\frac{1}{3}-\frac{5}{6}J\right)\frac{dy'}{dJ}-\frac{y'}{144}=0 \qquad (14)$$

Here

$$\alpha=-\frac{1}{12}, \quad \beta=-\frac{1}{12}, \quad \gamma=\frac{1}{3}$$

$$y_1'=J^{\frac{1}{12}}F\left(-\frac{1}{12}, \frac{7}{12}, \frac{1}{2}, \frac{J-1}{J}\right)$$

$$y_2'=J^{\frac{1}{12}}\sqrt{\frac{J-1}{J}}F\left(\frac{5}{12}, \frac{13}{12}, \frac{3}{2}, \frac{J-1}{J}\right)$$

Evaluating the constants as before we find from (4) and remembering that

$$F\left(\alpha, \beta, \gamma, \frac{J-1}{J}\right)=F\left(\beta, \alpha, \gamma, \frac{J-1}{J}\right)$$

exactly the same value for $\eta\ g_2^{\frac{1}{4}}$ as in (13). In Mathy's expression $-\frac{1}{12}$ appears in place of $\frac{7}{12}$, and $\frac{5}{12}$ in place of $\frac{13}{12}$.

Substituting from (10) and (13) in (1) the corrected expression for the mutual inductance becomes

$$M=4\pi\sqrt{2r_1}\Bigg[\frac{e_1}{2}\Bigg\{g_2^{-\frac{1}{4}}A\sqrt{2}\,.\,F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \frac{J-1}{J}\right)$$
$$-\frac{g_2^{-\frac{1}{4}}B}{\sqrt{6}}\sqrt{\frac{J-1}{J}}\,.\,F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}, \frac{J-1}{J}\right)\Bigg\}-\Bigg\{\frac{g_2^{\frac{1}{4}}B}{\sqrt{2}}F\left(-\frac{1}{12}, \frac{7}{12}, \frac{1}{2}, \frac{J-1}{J}\right)$$
$$+\frac{g_2^{\frac{1}{4}}A}{6\sqrt{6}}\sqrt{\frac{J-1}{J}}\,.\,F\left(\frac{5}{12}, \frac{13}{12}, \frac{3}{2}, \frac{J-1}{J}\right)\Bigg\}\Bigg]$$

Putting $x^2 = b^2 + r^2 + r_1^2$ we find

$$e_1 = \frac{x^2}{3r_1}, \quad e_2 = -\frac{x^2 - 6rr_1}{6r_1}, \quad e_3 = -\frac{x^2 + 6rr_1}{6r_1}$$

$$g_2^{\frac{1}{2}} = 2(e_1^2 + e_2^2 + e_3^2) = x^4 + 12r^2r_1^2$$

$$g_3 = 4e_1e_2e_3 = \frac{x^2(x^4 - 36r^2r_1^2)}{27r_1^3}$$

$$\frac{J-1}{J} = \frac{27g_3^2}{g_2^3} = \frac{\left[1 - 36\left(\frac{rr_1}{x^2}\right)^2\right]^2}{\left[1 + 12\left(\frac{rr_1}{x^2}\right)^2\right]^3} \tag{15}$$

The final expression for M becomes, therefore,

$$\left.\begin{aligned} \frac{M}{4\pi} = \Bigg[& \frac{x^2}{(x^4 + 12r^2r_1^2)^{\frac{1}{4}}} \left\{ \frac{A}{3^{\frac{3}{4}}} F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \frac{J-1}{J}\right) \right. \\ & \left. - \frac{B}{6.3^{\frac{1}{4}}} \sqrt{\frac{J-1}{J}} \, . F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}, \frac{J-1}{J}\right) \right\} \\ & - (x^4 + 12r^2r_1^2)^{\frac{1}{4}} \left\{ \frac{B}{3^{\frac{1}{4}}} F\left(-\frac{1}{12}, \frac{7}{12}, \frac{1}{2}, \frac{J-1}{J}\right) \right. \\ & \left. + \frac{A}{6.3^{\frac{3}{4}}} \sqrt{\frac{J-1}{J}} \, . F\left(\frac{5}{12}, \frac{13}{12}, \frac{3}{2}, \frac{J-1}{J}\right) \right\} \Bigg] \end{aligned}\right\} \tag{16}$$

where

$$\left.\begin{aligned} x^2 &= b^2 + r^2 + r_1^2 \\ A &= 1.311\ 028\ 777 \ldots \\ B &= 0.599\ 070\ 117 \ldots \\ \sqrt{\frac{J-1}{J}} &= \frac{1 - 36\left(\frac{rr_1}{x^2}\right)^2}{\left[1 + 12\left(\frac{rr_1}{x^2}\right)^2\right]^{\frac{3}{2}}} \end{aligned}\right\} \tag{17}$$

and

$$F(\alpha, \beta, \gamma, z) = 1 + \frac{\alpha\beta}{1.\gamma} z + \frac{\alpha(\alpha+1).\beta(\beta+1)}{1.2.\gamma(\gamma+1)} z^2 + \frac{\alpha(\alpha+1)(\alpha+2).\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)} . z^3$$

The formula for M is by no means so formidable to use as might be expected from its appearance, since the constants which enter

and the coefficients in the hypergeometrical series may be calculated once for all. Using seven-place logarithms, the values come out

$$\log\frac{A}{3^{\frac{3}{4}}}=9.759\ 7712 \qquad \log\frac{A}{6.3^{\frac{3}{4}}}=8.981\ 6199$$

$$\log\frac{B}{3^{\frac{1}{4}}}=9.658\ 1974 \qquad \log\frac{B}{6.3^{\frac{1}{4}}}=8.880\ 0461$$

and the coefficients a_1, a_2, a_3 of z, z^2, and z^3 in the series are given in the following table:

Series	a_1	a_2	a_3
$F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \frac{J-1}{J}\right)$	0.069 4444	0.035 5260	0.023 8485
$F\left(-\frac{1}{12}, \frac{7}{12}, \frac{1}{2}, \frac{J-1}{J}\right)$	−0.097 2222	−0.047 0358	−0.031 0523
$F\left(\frac{5}{12}, \frac{13}{12}, \frac{3}{2}, \frac{J-1}{J}\right)$	0.300 9259	0.177 6300	0.126 0562
$F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}, \frac{J-1}{J}\right)$	0.356 4814	0.216 3645	0.155 2615

From the magnitudes of these coefficients it is evident that the practical use of the formula (16) will be restricted to those cases for which $z=\frac{J-1}{J}$ is not much greater than about 0.2, and for the most precise work it should be still smaller. It is, therefore, more special than the formulæ of Maxwell and Nagaoka.

For the case $z=0$, each of the four series becomes equal to unity, and we have

$$\frac{M}{4\pi}=\frac{A}{3^{\frac{3}{4}}}\frac{x^2}{(x^4+12r^2r_1^{\,2})^{\frac{1}{4}}}-\frac{B}{3^{\frac{1}{4}}}(x^4+12r^2r_1^{\,2})^{\frac{1}{4}}$$

which, remembering that in this case $x^4=36r^2r_1^{\,2}$, reduces to the remarkably simple form

$$M=4\pi\sqrt{rr_1}(A-2B)=4\pi(0.112\ 888\ 543)\sqrt{rr_1}=1.418\ 599\sqrt{rr_1} \qquad (18)$$

If we introduce the distances

$$R_1=\sqrt{(r+r_1)^2+b^2}$$

$$R_2=\sqrt{(r-r_1)^2+b^2}$$

into equation (15) we find that the condition $z=0$ is equivalent to

$$R_1^2 = 2R_2^2 \tag{19}$$

That is, in all cases where the greatest distance between the circumferences of the two circles is $\sqrt{2}$ times the shortest distance between them, the mutual inductance is given by (18).

The following pairs of circles satisfying equation (19) are tabulated to aid in interpreting this equation:

$\frac{r_1}{r}$	$\frac{b}{r}$	
0.1716 $= 3 - 2\sqrt{2}$	0	Circles in the same plane.
0.2	0.4000	
0.3	0.8426	
0.4	1.1135	
0.5	1.3229	
0.6	1.4967	
0.7	1.6462	
0.8	1.7776	
0.9	1.8947	
1.0	2.0000	Equal circles.

Numerical results

Example 1. Formula (18): $r_1 = 25, \quad r = 25, \quad b = 50$

Therefore $$z = \frac{J-1}{J} = 0 \quad \text{(see table)} \; \sqrt{rr_1} = 25$$

$$\frac{M}{4\pi} = 25 \times 0.112\ 888\ 543 \ldots$$

$$= 2.822\ 2136 \ldots$$

By Nagaoka's formula $$\frac{M}{4\pi} = 2.822\ 213$$

By Maxwell's formula $$\frac{M}{4\pi} = 2.822\ 200$$

The values by the latter two formulæ are those found using seven place logarithms. The value from (18) may be carried out without difficulty to as many places of decimals as desired.

Example 2. Formula (16).

$$r_1 = 25 \qquad r = 20 \qquad b = 40$$

$$x^2 = 2625 \quad \frac{rr_1}{x^2} = \frac{4}{21} \qquad x^4 + 12r^2r_1^2 = 9\ 890\ 625$$

$$\frac{1}{4}\log(x^4 + 12r^2r_1^2) = 1.\ 748\ 8059$$

$$\log\frac{x^2}{(x^4+12r^2r_1^2)^{\frac{1}{4}}} = 1.\ 670\ 3234$$

$$1 - 36\left(\frac{rr_1}{x^2}\right)^2 = -0.\ 306\ \dot{1}225$$

$$1 + 12\left(\frac{rr_1}{x^2}\right)^2 = 1.\ 435\ 3742$$

$$\log\sqrt{\frac{J-1}{J}} = 9.\ 250\ 4476n$$

$$\log z = 8.\ 500\ 8952$$

	$F\left(\frac{1}{12},\frac{5}{12},\frac{1}{2},z\right)$	$F\left(\frac{7}{12},\frac{11}{12},\frac{3}{2},z\right)$	$F\left(-\frac{1}{12},\frac{7}{12},\frac{1}{2},z\right)$	$F\left(\frac{5}{12},\frac{13}{12},\frac{3}{2},z\right)$
	1.000 0000	1.000 0000	1.000 0000	1.000 0000
	0.002 2006	0.011 2962	−0.003 0808	0.009 5357
	0.000 0357	0.000 2173	−0.000 0473	0.000 1784
	0.000 0008	0.000 0049	−0.000 0010	0.000 0040
	1.002 2371	1.011 5184	0.996 8709	1.009 7181
$\log F$	= 0.000 9704	0.004 9738	9.998 6389	0.004 2002
log const.	= 9.759 7712	8.880 0461	9.658 1974	8.981 6199
$\log\frac{\sqrt{J-1}}{J}$	=	9.250 4476n	....	9.250 4476n
$\log(x^4+12r^2r_1^2)^{\frac{1}{4}}$	=	....	1.748 8059	1.748 8059
$\log\frac{x^2}{(x^4+12r^2r_1^2)^{\frac{1}{4}}}$	= 1.670 3234	1.670 3234	....	....
	1.431 0650	9.805 7909n	1.405 6422	9.985 0736n
	26.981 438	−0.639 427	25.447 327	−0.966 215
	−0.639 427		−0.966 215	
	27.620 865		24.481 112	

$$\therefore \frac{M}{4\pi} = 27.620\ 865 - 24.481\ 112 = 3.139\ 753\ cm$$

$$\text{By Nagaoka's formula } \frac{M}{4\pi} = 3.139\ 749$$

$$\text{By Maxwell's formula } \frac{M}{4\pi} = 3.139\ 766$$

For the more unfavorable case,

$$r_1 = 10,\ r = 5,\ b = 15,\ \therefore \log z = 8.562\ 0935$$

the values found were

By formula (16)	$\frac{M}{4\pi} = 0.62398\ cm$
By Nagaoka's formula	$= 0.62399$
By Maxwell's formula	$= 0.62400$

These examples will suffice to show that in those cases where the convergence is rapid the formula (16) gives the mutual inductance with precision. Before applying it in a given case, a preliminary rough calculation of z should be made to see if the convergence will be satisfactory. Because of the rather special applicability of the formula (16) it can not be regarded as in any way superseding the elliptic integral formulæ of Maxwell or the q series formulæ of Nagaoka.

The formula (18) gives, however, an exceedingly rapid and simple means for checking new formulæ, and should find extensive application in cases where the choice of the dimensions of the circles and their distance apart may be made to conform to equation (19).

WASHINGTON, February 1, 1910.